Georgia Institute of Technology

Computing parametric weighted Ehrhart polynomials of smooth polytopes

https://arxiv.org/abs/2511.09744
https://sites.gatech.edu/weightedehrhart

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Collaborators

Figure: This is joint work with Juliet Whidden and Josephine Yu.

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An introduction to polytopes

A (integral) polytope $P \subseteq \mathbb{R}^d$ is defined as the convex hull of a finite set of points, equivalently, **the closed bounded intersection of finitely many half-spaces.** Equivalently, P is the solution set to $Ax \leq \mathbf{b}$ for some (integral) matrix $A \in \mathbb{Z}^{n \times d}$ and (integral) vector $\mathbf{b} \in \mathbb{Z}^n$.

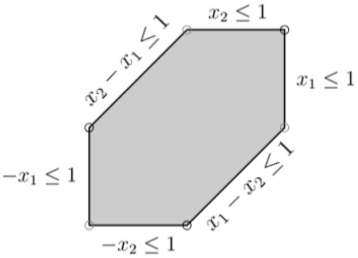

Figure: A example of an polytope. Modified from Joswig and Kulas [2010].

Ehrhart Theory (1)

"The central problem of Ehrhart Theory is counting the number of **integer points** in **integer dilates** of **integral polytopes**."
Integral polytopes are polytopes with all integral vertices.

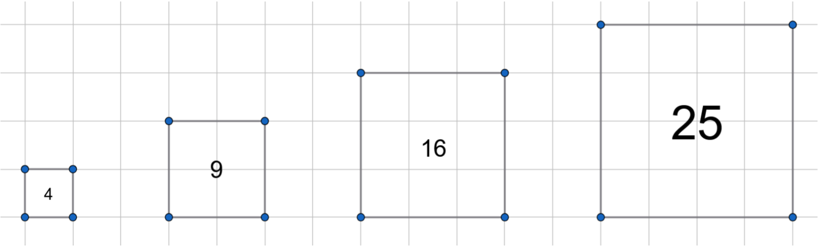

Figure: The first four integer dilates of the square P with vertices $(0,0),(0,1),(1,0),(1,1)$ and their corresponding integer point counts.
Question for the audience: In this case, what is $\#(tP \cap \mathbb{Z}^d)$?

Ehrhart Theory (2)

Theorem (Ehrhart Polynomial, Ehrhart [1967])

For any integral d-dimensional polytope P, $ehr_P(t) = \#(tP \cap \mathbb{Z}^d)$ is a degree d polynomial, which we refer to as the ***Ehrhart polynomial****.*

Definition (Ehrhart Series)

Moreover, if we encode the numbers $\mathrm{ehr}_P(0), \mathrm{ehr}_P(1), \mathrm{ehr}_P(2), \ldots$ as the coefficients of a power series, we get the **Ehrhart series**

$$\mathrm{Ehr}_P(z) = \sum_{t \geq 0} \mathrm{ehr}_P(t) z^t = \frac{h_d^* z^d + \ldots + h_0^*}{(1-z)^{d+1}}$$

We refer to the numerator $h_P^*(z) = h_d^* z^d + \ldots + h_0^*$ as the h^*-**polynomial**.

Ehrhart Theory (2.5)

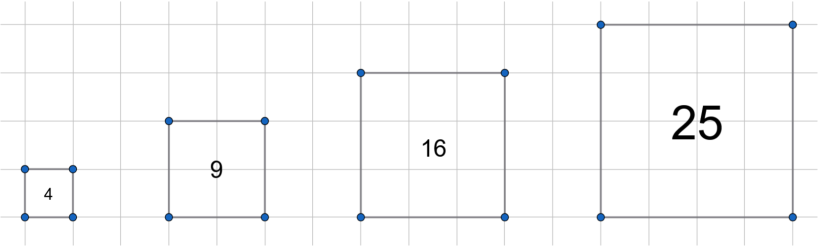

Figure: The Ehrhart polynomial of P in this case is $\mathbf{ehr}_P(t) = (t+1)^2$ and the corresponding Ehrhart series is $\mathbf{Ehr}_P(z) = 1 + 4z + 9z^2 + 16z^3 + 25z^4 + \ldots = (z+1)/(1-z)^3$.

Weighted Ehrhart Theory (1)

Now what happens if we "weigh" each of our points by a polynomial function?

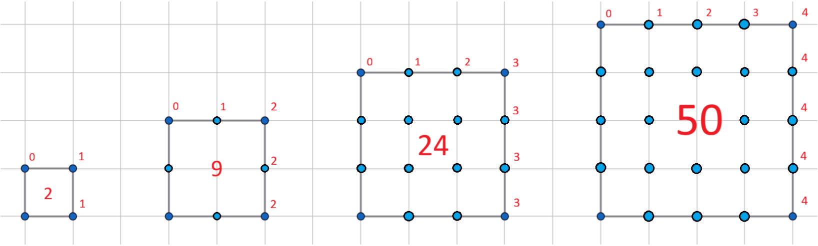

Figure: The first four integer dilates of the square P with vertices $(0,0),(0,1),(1,0),(1,1)$ and their corresponding **weighted** integer point counts according to the function $w = x_1$.

Weighted Ehrhart Theory (2)

Definition (Weighted Ehrhart Polynomial)

For any integral d-dimensional polytope P and homogeneous polynomial w of degree m, $\mathrm{ehr}_{P,w}(t) = \sum_{p \in tP \cap \mathbb{Z}^d} w(p)$ is a degree $d+m$ polynomial, called the **weighted Ehrhart polynomial**.

Definition (Weighted Ehrhart Series)

Moreover, if we encode $\mathrm{ehr}_{P,w}(0), \mathrm{ehr}_{P,w}(1), \mathrm{ehr}_{P,w}(2), \ldots$ as the coefficients of a power series, we get the **weighted Ehrhart series**

$$\mathrm{Ehr}_{P,w}(z) = \sum_{t \geq 0} \mathrm{ehr}_{P,w}(t) z^t = \frac{h^*_{d+m} z^{d+m} + \ldots + h^*_0}{(1-z)^{d+m+1}}$$

We refer to the numerator $h^*_{P,w}(z) = h^*_{d+m} z^{d+m} + \ldots + h^*_0$ as the **weighted h^*-polynomial**.

Weighted Ehrhart Theory (2.5)

Now what happens if we “weigh" each of our points by a polynomial function?

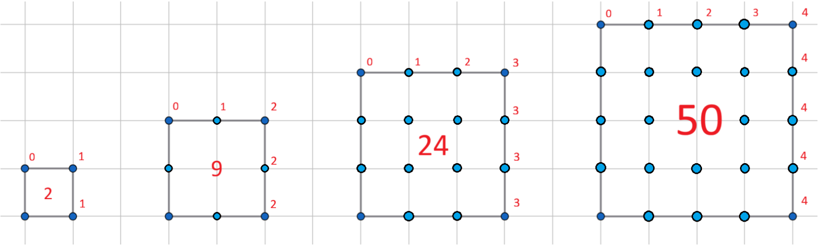

Figure: The weighted Ehrhart polynomial of P and w in this case is $\mathbf{ehr}_{P,w}(t) = (t+1)\sum_{x_1=0}^{t} x_1 = \frac{1}{2}t(t+1)^2$

and the corresponding weighted Ehrhart series is
$\mathbf{Ehr}_{P,w}(z) = 2z + 9z^2 + 24z^3 + 50z^4 + \ldots = (z^2+2z)/(1-z)^4$.

Weighted Ehrhart Theory (3)

Theorem (Stanley [1980])

For any d-dimensional integral polytope P, $h^*_P(z)$ has nonnegative integer coefficients.

However, this is not true in the weighted case! Not even for specialized cases of polytopes and weights!

Main Motivating Question

Which combinations of integral polytopes P and homogeneous weight functions w have weighted h^*-polynomials $h^*_{P,w}$ with nonnegative coefficients?

This is a difficult question for which we do not expect simple answers, so we will restrict to computing the weighted Ehrhart and h^*-polynomials for special parameterized families of polytopes.

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Parameterized Polytopes

Definition ($P_A(\mathbf{b})$)

Let A be a fixed $n \times d$ integer matrix. We will consider polytopes of the form

$$P_A(\mathbf{b}) = \{x \in \mathbb{R}^d | Ax \leq \mathbf{b}\}$$

for integral $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$.

We now ask the following motivating questions:

Further Motivating Questions

- **How does the weighted Ehrhart polynomial $\mathbf{ehr}_{P_A(\mathbf{b}),w}(t)$ and weighted h^*-polynomial $h^*_{P_A(\mathbf{b}),w}(z)$ depend on b?**
- Can we efficiently compute these polynomials? (Brandenburg et al. [2023] investigated this question extensively for unweighted Ehrhart polynomials of type A alcoved polytopes)

Piecewise Polynomiality

Luckily, we have an answer to the first question!

Corollary (Pukhlikov and Khovanskiĭ [1992a])

For a fixed $n \times d$ integer matrix A, the weighted integer point counting function $\mathbf{b} \to \sum_{p \in P_A(\mathbf{b}) \cap \mathbb{Z}^d} w(p)$ is a piecewise polynomial in terms of $b_1, \ldots, b_n$ with the pieces corresponding to different "combinatorial types" of $P_A(\mathbf{b})$.

Corollary

The weighted Ehrhart polynomial

$$ehr_{P_A(\mathbf{b}),w}(t) = \sum_{p \in P_A(t\mathbf{b}) \cap \mathbb{Z}^d} w(p)$$

and the weighted h^*-polynomial $h^*_{P_A(\mathbf{b}),w}(z)$ are piecewise polynomials in terms of t/z and $b_1, \ldots, b_n$.

Motion Of The Walls

Let's elaborate what we mean by "combinatorial type."

Definition (Motion Of The Walls, Pukhlikov and Khovanskiĭ [1992a])

A polytope P' is obtained as a **motion of the walls** from P if the normal fan of P' is a coarsening of the normal fan of P.

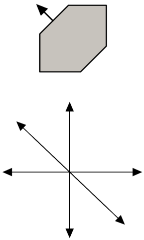

Figure: An example of P' (top right) obtained by a motion of the walls of P (top left).

Type Cone (1)

The set of polytopes belonging to the "same combinatorial type" can be formally described by the "type cone."

Definition (Type Cone)

For a polytope $P = P_A(\mathbf{b}^0)$ defined by $Ax \leq \mathbf{b}^0$, the set of all $\mathbf{b} \in \mathbb{Z}^n$ such that $P_A(\mathbf{b})$ is obtained from P by a motion of the walls is a polyhedral cone, known as the **type cone**. It is equivalently a secondary cone of the vector configuration consisting of the rows of A.

Corollary (Pukhlikov and Khovanskiĭ [1992a])

For a fixed $n \times d$ integer matrix A, the weighted integer point counting function $\mathbf{b} \to \sum_{p \in P_A(\mathbf{b}) \cap \mathbb{Z}^d} w(p)$ is a piecewise polynomial in terms of $b_1, \ldots, b_n$ with the pieces corresponding to the different type cones of $P_A(\mathbf{b})$.

Type Cone (2) - Alcoved Polytopes

Type cones allow us to distinguish different types of polytopes. Example: alcoved polytopes defined by the inequalities $x_i - x_j \leq a_{ij}$ for all $i, j \in [d+1]$ (note we use a_{ij} for alcoved polytopes) such that $a_{ij} + a_{jk} \geq a_{ik}$ and $x_{d+1} = 0$.

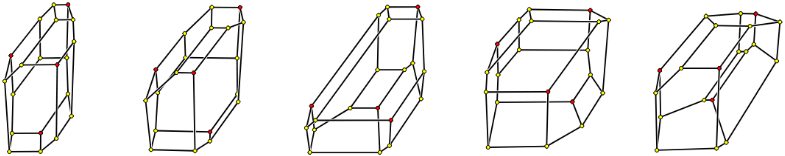

Figure: 5 types of 3D alcoved polytopes. Taken from Joswig and Kulas [2010].

Quick Recap

1. We consider parameterized polytopes of the form $P_A(\mathbf{b}) = \{x \in \mathbb{R}^d | Ax \leq \mathbf{b}\}$.
2. We can separate the set of all possible **b** into different **type cones**, which represent different types of polytopes.
3. Pukhlikov and Khovanskiĭ showed the weighted Ehrhart polynomial of $P_A(\mathbf{b})$ is a piecewise polynomial, with the pieces corresponding to the different type cones of $P_A(\mathbf{b})$.

Cool... can we actually compute these? **Yes.**

Our Main Result

We restrict to **smooth polytopes**.

Definition (Smooth Polytopes)

A d-dimensional integral polytope $P \subseteq \mathbb{R}^d$ is called **smooth** if each vertex cone of P is generated by a basis of $\mathbb{Z}^d$.

Theorem (H., Whidden, Yu (2025))

There exists an algorithm that when given a fixed smooth polytope $P_A(\mathbf{b}^0)$ and homogeneous weight function $w(\mathbf{x})$, explicitly computes the parametric weighted Ehrhart and h^*- polynomials for smooth polytopes $P_A(\mathbf{b})$ of the same type cone as $P_A(\mathbf{b}^0)$ in terms of the "right-hand side variables" b_i of P.

Unweighted versions of these results were obtained by De Loera and Sturmfels [2003] and Brandenburg et al. [2023].

Theorem (Pukhlikov and Khovanskiĭ [1992b], H., Whidden, Yu (2025))

For a smooth polytope $P_A(\mathbf{b})$ s.t. $\mathbf{b}$ is in the interior of a type cone,

$$\sum_{p \in P_A(\mathbf{b}) \cap \mathbb{Z}^d} w(p) = \left(Todd_{\mathbf{h}} \int_{P_A(\mathbf{b}+\mathbf{h})} w(y) dy \right) \Bigg|_{\mathbf{h}=0}$$

where $P_A(\mathbf{b}+\mathbf{h})$ is a shifted version of the polytope $P_A(\mathbf{b})$ for some small vector $\mathbf{h} \in \mathbb{Z}^n$.

The Todd Operator

Definition (Todd Operator)

The univariate Todd operator is a differential operator defined in terms of the Bernoulli numbers B_k:

$$\mathrm{Todd}_h = \sum_{k \geq 0} (-1)^k \frac{B_k}{k!} \left(\frac{d}{dh} \right)^k$$

The multivariate Todd operator is just the composition of the univariate Todd operators, i.e, $\mathrm{Todd}_{\mathbf{h}} = \prod \mathrm{Todd}_{h_i}$.

Quick Example

$\mathrm{Todd}_h(h^2) = h^2 + h + \frac{1}{6}$

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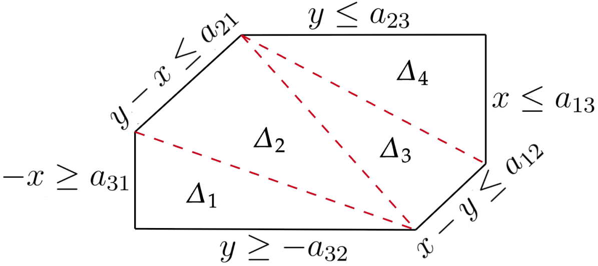
Example Run (1)

Let's define the 2D alcoved polytope by the inequalities

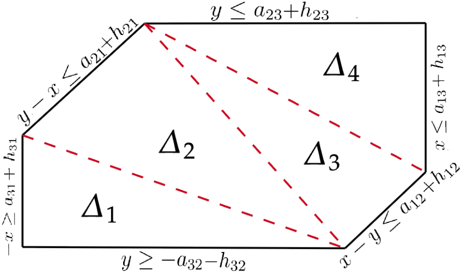
$$x - y \leq a_{12}$$
$$x \leq a_{13}$$
$$y - x \leq a_{21}$$
$$y \leq a_{23}$$
$$-x \leq a_{31}$$
$$-y \leq a_{32}$$



In the 2D alcoved case, there is only one maximal type.
Our goal is to describe the weighted integer count function (and thus weighted Ehrhart polynomial) as a ~~piecewise~~ polynomial in terms of the a_{ij}s.

Example Run (2)

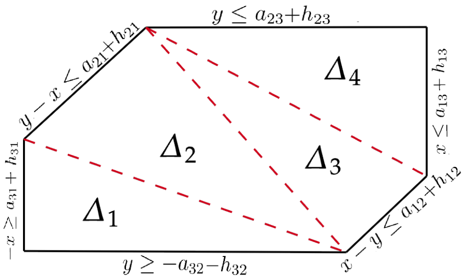
Our algorithm is as follows!



Step 1: Triangulate $P_A(\mathbf{b}^0)$; triangulation matches $P_A(\mathbf{b}+\mathbf{h})$ for all $\mathbf{b}+\mathbf{h}$ part of the same type cone as $\mathbf{b}^0$!

Recall $P_A(\mathbf{b}+\mathbf{h})$ is a shifted version of the polytope $P_A(\mathbf{b})$ for some small vector $\mathbf{h} \in \mathbb{Z}^n$.

Example Run (3)



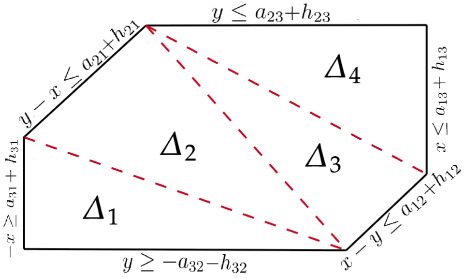
Step 2: For each simplex Δ, use the following theorem to compute the integral of w over Δ. Sum together these integrals to get the integral of w over $P_A(\mathbf{b}+\mathbf{h})$.

Theorem (Corollary 20 of Baldoni et al. [2011])

Let w be a homogeneous polynomial of degree m in d variables and $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1}$ be the vertices of a d-dimensional simplex Δ. Then

$$\int_\Delta w(\mathbf{x})d\mathbf{x} = \frac{vol(\Delta)}{2^m m! \binom{m+d}{m}} \sum_{1\le i_1\le\ldots\le i_m\le d+1} \sum_{\epsilon\in\{\pm1\}^m} \epsilon_1\epsilon_2\ldots\epsilon_m w\left(\sum_{k=1}^m \epsilon_k \mathbf{s}_{i_k}\right)$$

Example Run (4)



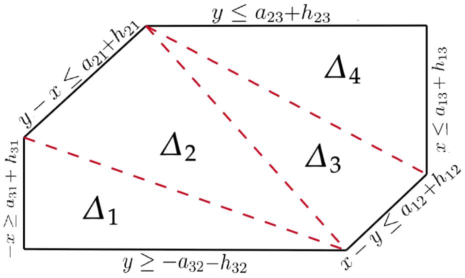
Step 3: Apply the multivariate Todd operator and evaluate at $\mathbf{h} = 0$ to get the weighted integer point count of $P_A(\mathbf{b})$ by Weighted K-P Theorem.

Theorem (Pukhlikov and Khovanskiĭ [1992b], H., Whidden, Yu (2025))

For smooth polytope $P_A(\mathbf{b})$ s.t. $\mathbf{b}$ is in the interior of a type cone,

$$\sum_{p \in P_A(\mathbf{b}) \cap \mathbb{Z}^d} w(p) = \left(Todd_{\mathbf{h}} \int_{P_A(\mathbf{b}+\mathbf{h})} w(y) dy \right) \Bigg|_{\mathbf{h}=0}$$

Example Run (5)



Step 4: The weighted Ehrhart polynomial of $P_A(\mathbf{b})$ is obtained by evaluating our weighted integer point count at $\mathbf{b}_i = t\mathbf{b}_i$ (and h^*-polynomial by a linear transformation of coordinates).

For example output and our code, please see our website:

`https://sites.gatech.edu/weightedehrhart`

Summary

- We are interested in computing **weighted Ehrhart polynomials** for parameterized polytopes $P_A(\mathbf{b})$, with the parameters $\mathbf{b}$ corresponding to the facets of $P_A(\mathbf{b})$.
- Pukhlikov and Khovanskiĭ tell us these are piecewise polynomials in terms of $\mathbf{b}$, with the pieces corresponding to the different type cones of $P_A(\mathbf{b})$.
- Moreover, if $P_A(\mathbf{b})$ is **smooth**, we can explicitly compute these weighted Ehrhart polynomials by using our Weighted Khovanskiĭ-Pukhlikov Theorem.
- These parameterized formulas for the weighted Ehrhart polynomial allow us to investigate the space of $h^*_{P,w}$ for various families of smooth polytopes P.

Michael Joswig and Katja Kulas. Tropical and ordinary convexity combined, 2010. URL https://arxiv.org/abs/0801.4835.

E. Ehrhart. Sur une problème de géométrie diophantine linéaire. *Journal für die reine und angewandte Mathematik*, 1967(227): 25–49, 1967. doi: https://doi.org/10.1515/crll.1967.227.25.

Richard P. Stanley. Decompositions of rational convex polytopes. In J. Srivastava, editor, *Combinatorial Mathematics, Optimal Designs and Their Applications*, volume 6 of *Annals of Discrete Mathematics*, pages 333–342. Elsevier, 1980. doi: https://doi.org/10.1016/S0167-5060(08)70717-9. URL https://www.sciencedirect.com/science/article/pii/S0167506008707179.

Marie-Charlotte Brandenburg, Sophia Elia, and Leon Zhang. Multivariate volume, ehrhart, and h∗-polynomials of polytropes. *Journal of Symbolic Computation*, 114:209–230, January 2023. doi: 10.1016/j.jsc.2022.04.011.

A. V. Pukhlikov and A. G. Khovanskiĭ. Finitely additive measures of virtual polyhedra. *Algebra i Analiz*, 4(2):161–185, 1992a. URL https://www.math.toronto.edu/askold/1992-Alg-An-2-english.pdf.

Jesús A. De Loera and Bernd Sturmfels. Algebraic unimodular counting. *Math. Program.*, 96(2):183–203, 2003. doi: 10.1007/s10107-003-0383-9. Algebraic and geometric methods in discrete optimization.

A. V. Pukhlikov and A. G. Khovanskiĭ. The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes. *Algebra i Analiz*, 4(4):188–216, 1992b. URL https://www.math.toronto.edu/askold/1992-Alg-An-4-english.pdf.

Velleda Baldoni, Nicole Berline, Jesus A. De Loera, Matthias Köppe, and Michele Vergne. How to integrate a polynomial over a simplex. *Math. Comp.*, 80(273):297–325, 2011. doi: 10.1090/S0025-5718-2010-02378-6.

Thank you!

https://arxiv.org/abs/2511.09744
https://sites.gatech.edu/weightedehrhart

Computational Complexity (1)

Here, we briefly discuss the computational complexity of our algorithm.

Theorem (H., Whidden, Yu (2025))

Our main result is that for a fixed dimension d, a smooth polytope $P_A(\mathbf{b}^0) \subseteq \mathbb{R}^d$ with n facets and a monomial weight w of degree m, our algorithm takes $O(n^{d^2/4} \cdot 2^m \binom{d+m}{d} \cdot (d+m))$ time to complete.

- $O(n^{d^2/4})$ is the upper bound on the number of simplices in the triangulation of $P_A(\mathbf{b}^0)$.
- $O(2^m \binom{d+m}{d})$ is the upper bound on the number of terms from evaluating Corollary 20 from Baldoni et al. [2011] on each simplex.
- The Todd operator takes at most $O(d+m)$ time to run on each term.

Computational Complexity (2)

Here, we briefly discuss the computational complexity of our algorithm.

Theorem (H., Whidden, Yu (2025))

Our main result is that for a fixed dimension d, a smooth polytope $P_A(\mathbf{b}^0) \subseteq \mathbb{R}^d$ with n facets and a monomial weight w of degree m, our algorithm takes $O(n^{d^2/4} \cdot 2^m \binom{d+m}{d} \cdot (d+m))$ time to complete.

We note that our algorithm is exponential in terms of m, however, if we assume that $m = o(n)$, where n is the number of facets, then our algorithm is significantly faster than naive interpolation, which solves for our desired degree $d+m$ polynomial in n variables by solving a system of $\binom{n+d+m}{n}$ equations in terms of all possible $\binom{n+d+m}{n}$ monomials of degree at most $d+m$.

Non-convexity (1)

For a fixed polytope $P_A(\mathbf{b}^0)$, we observe that if $h^*_{P_A(\mathbf{b}^0),w_1}$ and $h^*_{P_A(\mathbf{b}^0),w_2}$ have nonnegative coefficients, then any convex combination of the weights $w = cw_1 + (1-c)w_2$ will also result in $h^*_{P_A(b),w}$ having nonnegative coefficients.

Moreover, the set of all weights w giving rise to a weighted h^*-polynomial with nonnegative coefficients forms a convex cone.

However, for a fixed weight w, the set of $P_A(\mathbf{b})$ such that $h^*_{P_A(\mathbf{b}),w}$ has nonnegative coefficients is not convex.

Non-convexity (2)

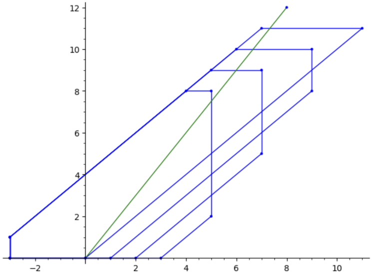


Figure: From left to right, the four blue polygons show P_0, P_1, P_2, P_3. The green line is the zero locus of the linear weight $w = 2y - 3x$.

Let $\mathbf{v} = (3, 5, 4, 8, 3, 0)$, $\mathbf{h} = (-1, 2, 0, 1, 0, 0)$, and $P_i = P_A(\mathbf{v} + i\mathbf{h})$, where $P_A(\mathbf{b})$ is the two-dimensional alcoved polytope described earlier in our Example Run.

Then for $w = 2y - 3x$, we have that

$$h^*_{P_0,w} = 10z^3 + 65z^2 + 25z$$
$$h^*_{P_1,w} = -10z^3 - 37z^2 - 7z$$
$$h^*_{P_2,w} = -9z^3 - 39z^2 - 10z$$
$$h^*_{P_3,w} = 15z^3 + 67z^2 + 18z$$

thus showing the set of all polytopes P that give nonnegative h^*-polynomial coefficients is not convex, even for a fixed linear weight w.

Sign patterns

Let $L_{P,m}$ be the space of all $h^*_{P,w}$ polynomials for all homogeneous weights w of degree m, which is the image of the map $w \to h^*_{P,w}$.

We expect this space to be full-dimensional when the dimension of the set of homogeneous weights w of degree m (which is $\binom{d+m-1}{m}$) is at least the maximum dimension of $L_{P,w}$ (which is $d+m$).

This holds except when $d = 2$ or $m = 1$, which led us to ask if all sign patterns were possible in those cases. We were able to computationally verify all sign patterns of weighted h^*-polynomials of alcoved polytopes for dimension 2, degrees 1 to 5 and dimension 3, degree 1. Some sign patterns were especially rare.

Large dilates

Theorem (H., Whidden, Yu (2025))

The roots of $h^*_{nP,w}$ converge towards the roots of the Eulerian polynomial $A_{d+m}(z)$ as $n \to \infty$. Moreover, the coefficients of $h^*_{nP,w}$ eventually have the same sign for large enough n.

Proof is similar to the unweighted case.

However, when we tried to compute a universal bound $N_{w,d}$ for which the coefficients of $h^*_{nP,w}$ had the same sign for all $n \geq N_{w,d}$, we found counterexamples where a large dilate was required to get all coefficients the same sign.

Note: The unweighted version of this observation was brought to us by Ben Braun.